A characterization of classical orthogonal Laurent polynomials

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ABSTRACT

In [3] certain Laurent polynomials of ${}_2F_1$ genus were called "Jacobi Laurent polynomials". These Laurent polynomials belong to systems which are orthogonal with respect to a moment sequence $((a)_n/(c)_n)_{n\in\mathbb{Z}}$ where a, c are certain real numbers. Together with their confluent forms, belonging to systems which are orthogonal with respect to $(1/(c)_n)_{n\in\mathbb{Z}}$ respectively $((a)_n)_{n\in\mathbb{Z}}$, these Laurent polynomials will be called "classical". The main purpose of this paper is to determine all the simple (see section 1) orthogonal systems of Laurent polynomials of which the members satisfy certain second order differential equations with polynomial coefficients, analogously to the well known characterization of S. Bochner [1] for ordinary polynomials.

1. INTRODUCTION

In 1929 Bochner [1] gave in fact the nowadays well known characterization of the "classical" orthogonal polynomials by means of second order differential equations with polynomial coefficients. See also [2, p. 150]. The main purpose of the present paper is to give a similar characterization of a certain class of orthogonal systems of Laurent polynomials.

We only consider Laurent polynomials with real or complex coefficients. Throughout this paper the set of nonnegative integers is denoted by $\mathbb{N}_0$ and a, c, a^*, c^* are complex numbers. A sequence $(Q_n)_{n=0}^{\infty}$ of Laurent polynomials will be called *simple* if

$$Q_{2n}(x) = \alpha_{-n}^{(2n)}x^{-n} + \alpha_{-n+1}^{(2n)}x^{-n+1} + \cdots + \alpha_{n-1}^{(2n)}x^{n-1} + \alpha_n^{(2n)}x^n$$

and

$$Q_{2n+1}(x) = \alpha_{-n-1}^{(2n+1)}x^{-n-1} + \alpha_{-n}^{(2n+1)}x^{-n} + \cdots + \alpha_{n-1}^{(2n+1)}x^{n-1} + \alpha_{n}^{(2n+1)}x^{n}$$

where $\alpha_{-n}^{(2n)}, \ldots, \alpha_{n}^{(2n)}, \alpha_{-n-1}^{(2n+1)}, \ldots, \alpha_{n}^{(2n+1)} \in \mathbb{C}$ such that $\alpha_{-n}^{(2n)} = \alpha_{-n-1}^{(2n+1)} = 1$ and $\alpha_{n}^{(2n)} \neq 0$, $\alpha_{n}^{(2n+1)} \neq 0$, $n = 0, 1, 2, \ldots$.

In [3, section 1] it has been shown that a simple system of Laurent polynomials $(Q_n)_{n=0}^{\infty}$ is orthogonal with respect to a moment functional $\mathscr{L}$ (i.e. $\mathscr{L}(Q_n^2) \neq 0$ and $\mathscr{L}(Q_n Q_k) = 0$ if $k \neq n$, $k, n = 0, 1, \ldots$) if and only if there exists non zero numbers f_n and g_n such that

$$(1.1) \qquad \begin{cases} Q_{2n+1}(x) = (x^{-1} + g_{2n+1})Q_{2n}(x) + f_{2n+1}Q_{2n-1}(x), \\ Q_{2n+2}(x) = (1 + g_{2n+2}x)Q_{2n+1}(x) + f_{2n+2}Q_{2n}(x) \end{cases} \qquad n = 0, 1, 2, \ldots$$

with $Q_{-1}(x) = 0$ and $Q_0(x) = 1$.

If $Q_{2n}(x) = x^{-n}V_{2n}(x)$, $Q_{2n+1}(x) = x^{-n-1}V_{2n+1}(x)$, then (1.1) is equivalent to

$$(1.2) \qquad V_n(x) = (1 + g_n x)V_{n-1}(x) + f_n x V_{n-2}(x) \quad n = 1, 2, \ldots$$

with $V_{-1}(x) = 0$ and $V_0(x) = 1$.

The following examples 1-3 of simple orthogonal systems of Laurent polynomials can be found in [3].

EXAMPLE 1.

$$\begin{aligned} B_{2n}(x) &= x^{-n}{}_2F_1(-2n, -a; -c-2n+1; x) \\ B_{2n+1}(x) &= x^{-n-1}{}_2F_1(-2n-1, -a; -c-2n; x) \end{aligned} \qquad n = 0, 1, \ldots,$$

with moment functional $\mathscr{L}$ given by

$$\mathscr{L}(x^n) = \frac{(a)_{-n}}{(c)_{-n}}, \quad n \in \mathbb{Z}, \ (a, -c, a-c \notin \mathbb{N}_0).$$

EXAMPLE 2.

$$\begin{aligned} R_{2n}(x) &= x^{-n}{}_1F_1(-2n; -c-2n+1; -x) \\ R_{2n+1}(x) &= x^{-n-1}{}_1F_1(-2n-1; -c-2n; -x) \end{aligned} \qquad n = 0, 1, \ldots,$$

with moment functional $\mathscr{L}$ given by

$$\mathscr{L}(x^n) = \frac{1}{(c)_{-n}}, \quad n \in \mathbb{Z}, \ (-c \notin \mathbb{N}_0).$$

EXAMPLE 3.

$$\begin{aligned} S_{2n}(x) &= x^{-n}{}_2F_0(-2n, -a; -x) \\ S_{2n+1}(x) &= x^{-n-1}{}_2F_0(-2n-1, -a; -x) \end{aligned} \qquad n = 0, 1, \ldots,$$

with moment functional $\mathscr{L}$ given by

$$\mathscr{L}(x^n) = (a)_{-n}, \ n \in \mathbb{Z}, \ (a \notin \mathbb{N}_0).$$

In this paper the (systems of) Laurent polynomials of examples 1, 2 and 3 will be called *classical*.

It is easily shown that for the classical Laurent polynomials we have the following differential equations

$$(1.3)\qquad x^2(1-x)B_{2n}''+x[(a-1)x-c+1]B_{2n}'-n[(a-n)x+c+n]B_{2n}=0,$$

$$(1.3')\qquad x^2(1-x)B_{2n+1}''+x[(a-2)x-c+2]B_{2n+1}' \\ -[n(a-n-1)x+(n+1)(c+n)]B_{2n+1}=0,$$

$$(1.4)\qquad x^2R_{2n}''+x(x-c+1)R_{2n}'-n(x+c+n)R_{2n}=0,$$

$$(1.4')\qquad x^2R_{2n+1}''+x(x-c+2)R_{2n+1}'-[nx+(n+1)(c+n)]R_{2n+1}=0,$$

$$(1.5)\qquad x^3S_{2n}''-x[(a-1)x-1]S_{2n}'+n[(a-n)x+1]S_{2n}=0,$$

$$(1.5')\qquad x^3S_{2n+1}''-x[(a-2)x-1]S_{2n+1}'+[n(a-n-1)x+n+1]S_{2n+1}=0,$$
$n=0,1,2,\ldots.$

EXAMPLE 4. $W_0(x)=1$

$$\left.\begin{aligned} W_{2n+1}(x)&=x^{-n-1}+px^n \quad n=0,1,\ldots\\ W_{2n}(x)&=x^{-n}+px^n \qquad n=1,2,\ldots\end{aligned}\right\}\quad p\in\mathbb{C}\setminus\{0\}.$$

Here we have the recurrence relations

$$\left.\begin{aligned} W_{2n+1}(x)&=(x^{-1}+1)W_{2n}(x)-W_{2n-1}(x)\\ W_{2n+2}(x)&=(1+x)W_{2n+1}(x)-W_{2n}(x)\end{aligned}\right\}\quad n=1,2,\ldots$$

with $W_0(x)=1$, $W_1(x)=x^{-1}+p$, $W_2(x)=x^{-1}+px$.

$(W_n)_{n=0}^{\infty}$ is orthogonal with respect to the moment functional $\mathscr{L}$ given by

$$\mathscr{L}(x^n)=\begin{cases}-p^{-1} & n=0,1,2,\ldots\\ 1 & n=-1,-2,\ldots\end{cases}.$$

For W_n we have the differential equations

$$(1.6)\qquad x^2W_{2n}''+xW_{2n}'-n^2W_{2n}=0,$$

$$(1.6')\qquad x^2W_{2n+1}''+2xW_{2n+1}'-n(n+1)W_{2n+1}=0.$$

In all the examples 1–4 the $(2n)$th Laurent polynomial X_{2n} and the $(2n+1)$th Laurent polynomial X_{2n+1} satisfy differential equations

$$\left.\begin{aligned}(1.7)\qquad & A_3X_{2n}''+A_2X_{2n}'+A_1^{(2n)}X_{2n}=0\\ (1.7')\qquad & A_3X_{2n+1}''+A_2^*X_{2n+1}'+A_1^{(2n+1)}X_{2n+1}=0\end{aligned}\right\}\quad n=0,1,2,\ldots$$

where A_3, A_2, A_2^* are fixed polynomials, independent of n while $A_3 \not\equiv 0$, $\deg A_2 = \deg A_2^*$ and $A_1^{(2n)}, A_1^{(2n+1)}$ are polynomials depending on n but with $\deg A_1^{(2n)} \leqq 1$ and $\deg A_1^{(2n+1)} \leqq 1$, $n = 0, 1, 2, \dots$.

In section 2 it will be shown that the systems in the examples 1–4 are essentially the only simple orthogonal systems of Laurent polynomials such that differential equations as in (1.7) and (1.7′) are satisfied.

2. Let $(Q_n)_{n=0}^{\infty}$ be a simple orthogonal system of Laurent polynomials such that

$$\begin{aligned} &(2.1) && A_3 Q_{2n}'' + A_2 Q_{2n}' + A_1^{(2n)} Q_{2n} = 0 \\ & && && n = 0, 1, 2, \dots \\ &(2.1') && A_3 Q_{2n+1}'' + A_2^* Q_{2n+1}' + A_1^{(2n+1)} Q_{2n+1} = 0 \end{aligned}$$

where A_3, A_2, A_2^* are fixed polynomials with $A_3 \not\equiv 0$, $\deg A_2 = \deg A_2^*$ and $A_1^{(2n)}$ and $A_1^{(2n+1)}$ are polynomials depending on n with $\deg A_1^{(2n)} \leqq 1$ and $\deg A_1^{(2n+1)} \leqq 1$.

Assume

$$Q_{2n}(x) = x^{-n} + \alpha_{-n+1}^{(2n)} x^{-n+1} + \dots + \alpha_n^{(2n)} x^n, \quad \alpha_n^{(2n)} \neq 0,$$

$$Q_{2n+1}(x) = x^{-n-1} + \alpha_{-n}^{(2n+1)} x^{-n} + \dots + \alpha_n^{(2n+1)} x^n, \quad \alpha_n^{(2n+1)} \neq 0,$$

$$n = 0, 1, 2, \dots.$$

Then $n = 0$ in (2.1′) gives

$$2A_3 - xA_2^* + x^2(1 + \alpha_0^{(1)} x) A_1^{(1)} = 0.$$

This implies $A_3(0) = 0$. With $A_3(x) = x a_2(x)$ we get moreover

$$(2.2) \qquad 2a_2(0) - A_2^*(0) = 0$$

and

$$(2.3) \qquad \deg(2a_2 - A_2^*) \leqq 3.$$

From (2.1) with $n = 1$ we get

$$2a_2 + (-1 + \alpha_1^{(2)} x^2) A_2 + x(1 + \alpha_0^{(2)} x + a_1^{(2)} x^2) A_1^{(2)} = 0.$$

This gives $2a_2(0) - A_2(0) = 0$, so by (2.2) we have

$$(2.4) \qquad A_2^*(0) = A_2(0).$$

Furthermore we see that $\deg(2a_2 + (-1 + \alpha_1^{(2)} x^2) A_2) \leqq 4$. Together with (2.3) this gives $\deg(A_2^* + (-1 + \alpha_1^{(2)} x^2) A_2) \leqq 4$. Since $\deg A_2^* = \deg A_2$ this implies

$$\deg^* A_2 = \deg A_2 \leqq 2.$$

Now it follows easily from (2.1) that

$$\deg A_3 \leqq 3.$$

Taking $n=1$ in (2.1′) we obtain

$$(6+2\alpha_{-1}^{(3)}x)a_2+(-2-\alpha_{-1}^{(3)}x+\alpha_1^{(3)}x^3)A_2^*+$$
$$+x(1+\alpha_{-1}^{(3)}x+\alpha_0^{(3)}x^2+\alpha_1^{(3)}x^3)A_1^{(3)}=0,$$

so $6a_2(0)-2A_2^*(0)=0$, and with (2.2) and (2.4) we get

$$a_2(0)=A_2(0)=A_2^*(0)=0.$$

Hence

$$A_3(x)=x^2a_1(x) \quad \text{with } \deg a_1\leqq 1,\ a_1\not\equiv 0$$

and

$$A_2(x)=x(\varrho x+\sigma), \quad A_2^*(x)=x(\varrho^*x+\sigma^*), \quad (\varrho,\sigma,\varrho^*,\sigma^*\in\mathbb{C}).$$

Now we have (essentially) the following possibilities for A_3

I $A_3(x)=x^2(\lambda-x)$, $(\lambda\in\mathbb{C},\ \lambda\neq 0)$

II $A_3(x)=x^2$

III $A_3(x)=x^3$.

I $A_3(x)=x^2(\lambda-x)$, $(\lambda\in\mathbb{C},\ \lambda\neq 0)$.

In this case we have

$$\begin{aligned}
&(2.5) && x^2(\lambda-x)Q_{2n}''+x(\varrho x+\sigma)Q_{2n}'+A_1^{(2n)}Q_{2n}=0, \\
&(2.5') && x^2(\lambda-x)Q_{2n+1}''+x(\varrho^*x+\sigma^*)Q_{2n+1}'+A_1^{(2n+1)}Q_{2n}=0
\end{aligned} \qquad n=0,1,2,\ldots.$$

Since $\lambda^nQ_{2n}(\lambda x)$ and $\lambda^{n+1}Q_{2n+1}(\lambda x)$ satisfy differential equations of the same form with $\lambda=1$ we may assume that indeed $\lambda=1$. Then equating to zero the leading and trailing coefficients in (2.5) and (2.5′) gives

$$\begin{aligned}
A_1^{(2n)}(x)&=-n[(\varrho-n+1)x-\sigma+n+1], \\
A_1^{(2n+1)}(x)&=-[n(\varrho^*-n+1)x+(n+1)(-\sigma^*+n+2)]
\end{aligned} \qquad n=0,1,2,\ldots.$$

With $\varrho=a-1$, $\sigma=-c+1$, $\varrho^*=a^*-2$, $\sigma^*=-c^*+2$ (2.5) and (2.5′) become

$$\begin{aligned}
&(2.6) && x^2(1-x)Q_{2n}''+x[(a-1)x-c+1]Q_{2n}'-n[(a-n)x+c+n]Q_{2n}=0, \\
&(2.6') && x^2(1-x)Q_{2n+1}''+x[(a^*-2)x-c^*+2]Q_{2n+1}' \\
& && \quad -[n(a^*-n-1)x+(n+1)(c^*+n)]Q_{2n+1}=0, \\
& && n=0,1,2,\ldots,
\end{aligned}$$

which are the same equations as in (1.3) and (1.3′) if $a=a^*$ and $c=c^*$. The ordinary polynomials $V_{2n}(x)=x^nQ_{2n}(x)$ and $V_{2n+1}(x)=x^{n+1}Q_{2n+1}(x)$ satisfy the hypergeometric differential equations

$$\begin{aligned}
&(2.7) && x(1-x)V_{2n}''+[(a+2n-1)x-(c+2n-1)]V_{2n}'-2naV_{2n}=0, \\
&(2.7') && x(1-x)V_{2n+1}''+[(a^*+2n)x-(c^*+2n)]V_{2n+1}'-(2n+1)a^*V_{2n+1}=0, \\
& && n=0,1,\ldots.
\end{aligned}$$

Clearly (2.6) and (2.6′) are equivalent to (2.7) and (2.7′), so we have to determine all the sequences of polynomials $(V_n)_{n=0}^{\infty}$ with $\deg V_n = n$ and $V_n(0)=1$ such that the corresponding sequence $(Q_n)_{n=0}^{\infty}$ is an orthogonal system of Laurent polynomials, and such that (2.7) and (2.7′) are satisfied. For this purpose we consider the differential equation

$$x(1-x)\,y_n'' + [(a+n-1)x-(c+n-1)]\,y_n' - nay_n = 0, \quad n \in \mathbb{N}_0;\ a, c \in \mathbb{C}. \tag{2.8}$$

A polynomial solution y_n of (2.8) will be called *proper* if $\deg y_n = n$ and $y_n(0)=1$, $(n \in \mathbb{N}_0)$.

For $a, c \in \mathbb{C}$ we determine all possible sequences $(y_{2n})_{n=0}^{\infty}$ respectively $(y_{2n+1})_{n=0}^{\infty}$ of proper solutions of (2.8).

When $y_n(x) = \sum_{k=0}^{n} \xi_k x^k$, $(\xi_k \in \mathbb{C},\ k=0,1,\dots,n)$, then y_n satisfies (2.8) if and only if

$$(-n+k)(-a+k)\xi_k = (1+k)(-c+1-n+k)\xi_{k+1} \quad \text{as } k=0,1,\dots,n-1. \tag{2.9}$$

We have the following possibilities for a and c.

(1) $a \notin \mathbb{N}_0$, $-c \notin \mathbb{N}_0$. The proper solutions of (2.8) are precisely

$$y_n(x) = {}_2F_1(-n,-a;-c-n+1;x) \quad n=0,1,\dots.$$

(2) $a \in \mathbb{N}_0$, $-c \notin \mathbb{N}_0$. It follows easily from (2.9) that for $n \geqq a+1$ there are no proper solutions of (2.8).

(3) $a \notin \mathbb{N}_0$, $-c \in \mathbb{N}_0$. As in the previous case, using (2.9) we see that there are no proper solutions of (2.8) if $n \geqq -c+1$.

(4) $a \in \mathbb{N}_0$, $-c \in \mathbb{N}_0$. In this case we distinguish between $a-c \geqq 3$ and $a-c \leqq 2$.

If $a-c \geqq 3$, then for $n=a-c$ and for $n=a-c-1$ we must have $0 \leqq c+n-1$ or $a+1 \leqq n$ and it follows from (2.9) that for these values of n

$$0 = \xi_0 = \cdots = \xi_{c+n-1} \quad \text{or} \quad \xi_{a+1} = \cdots = \xi_n = 0$$

if $y_n(x) = \sum_{k=0}^{n} \xi_k x^k$ is a solution of (2.8). Hence if $a-c \geqq 3$ then there do not exist sequences $(y_{2n})_{n=0}^{\infty}$ or $(y_{2n+1})_{n=0}^{\infty}$ of proper solutions of (2.8). If $a-c \leqq 2$ then we have the remaining six cases:

(i) $a=0$, $-c=0$. The proper solutions are precisely $y_0(x)=1$ and $y_n(x) = 1+\xi_n x^n$, $\xi_n \neq 0$, $n \geqq 1$.

(ii) $a=1$, $-c=0$. Then for $n=1$ the polynomial solutions of (2.8) are $y_1(x) = \xi_1 x$, $\xi_1 \in \mathbb{C}$, so for $n=1$ there is no proper solution. When $n \neq 1$ the proper solutions are $y_0(x)=1$ and $y_n(x) = 1-(n/(n-1))x+\xi_n x^n$, $\xi_n \neq 0$, $n \geqq 2$.

(iii) $a=0$, $-c=1$. Now for $n=1$ the polynomial solutions of (2.8) are constant, so there is no proper solution y_1. As $n \neq 1$ we have the proper solutions $y_0(x)=1$ and $y_n(x) = 1+\xi_n(-n/(n-1))x^{n-1}+x^n)$, $\xi_n \neq 0$, $n \geqq 2$.

(iv) $a=2$, $-c=0$. Here for $n=1$ and $n=2$ the polynomial solutions of (2.8) are $y_1(x)=\xi_1 x$ respectively $y_2(x)=\xi_2 x^2$, $\xi_1, \xi_2 \in \mathbb{C}$; so for $n=1,2$ there are no proper solutions of (2.8).

(v) $a=1,\ -c=1$. Now there is no proper solution of (2.8) if $n=2$, since for $n=2$ the polynomial solutions are $y_2(x)=\xi x,\ \xi\in\mathbb{C}$. When $n\neq 2$ the proper solutions are $y_0(x)=1,\ y_1(x)=1+x$ and $y_n(x)=1-(n/(n-2))x+\xi_n(-n/(n-2))x^{n-1}+ +x^n),\ \xi_n\neq 0,\ n\geqq 3$.

(vi) $a=0,\ -c=2$. Then for $n=1$ and $n=2$ the polynomial solutions of (2.8) are constant, so there are no proper solutions as $n=1$ or $n=2$.

We summarize the results with respect to the existence of sequences $(y_{2n})_{n=0}^{\infty}$, $(y_{2n+1})_{n=0}^{\infty}$ of proper solutions of (2.8) in the following table, which is exhaustive in the sense that it contains all the possible even or odd indexed sequences of proper solutions of (2.8).

Table 1

		$(y_{2n})_{n=0}^{\infty}$	$(y_{2n+1})_{n=0}^{\infty}$
(1)	$a\notin\mathbb{N}_0,\ -c\notin\mathbb{N}_0$	$y_{2n}={}_2F_1(-2n,-a;-c-2n+1;x)$, $n\geqq 0$	$y_{2n+1}={}_2F_1(-2n-1,-a;-c-2n;x)$, $n\geqq 0$
(2)	$a\in\mathbb{N}_0,\ -c\notin\mathbb{N}_0$	—	—
(3)	$a\notin\mathbb{N}_0,\ -c\in\mathbb{N}_0$	—	—
(4)	$a\in\mathbb{N}_0,\ -c\in\mathbb{N}_0$, $a-c\geqq 3$	—	—
(4,i)	$a=0,\ -c=0$	$y_0=1$ $y_{2n}=1+\xi_{2n}x^{2n},\ n\geqq 1$	$y_{2n+1}=1+\xi_{2n+1}x^{2n+1},\ n\geqq 0$
(4,ii)	$a=1,\ -c=0$	$y_0=1$ $y_{2n}=1-\dfrac{2n}{2n-1}x+\xi_{2n}x^{2n},\ n\geqq 1$	—
(4,iii)	$a=0,\ -c=1$	$y_0=1$ $y_{2n}=1+\xi_{2n}\left(\dfrac{-2n}{2n-1}x^{2n-1}+x^{2n}\right)$, $n\geqq 1$	—
(4,iv)	$a=2,\ -c=0$	—	—
(4,v)	$a=1,\ -c=1$	—	$y_1=1+x$ $y_{2n+1}=1-\dfrac{2n+1}{2n-1}x+\xi_{2n+1}\left(\dfrac{-2n-1}{2n-1}x^{2n}+x^{2n+1}\right)$, $n\geqq 1$
(4,vi)	$a=0,\ -c=2$	—	—

In this table all the ξ_k are non zero

Now we return to the equations (2.7) and (2.7′). It follows from table 1 that an even indexed sequence $(V_{2n})_{n=0}^{\infty}$ of proper solutions of (2.7) exists precisely when

(1) $a\notin\mathbb{N}_0,\ \ -c\notin\mathbb{N}_0$ or

(i) $a=0,\ \ -c=0$ or

(ii) $a=1,\ \ -c=0$ or

(iii) $a=0,\ \ -c=1$,

and that an odd indexed sequence $(V_{2n+1})_{n=0}^{\infty}$ of proper solutions of (2.7′) exists only if

(1*) $a^*\notin\mathbb{N}_0,\ -c^*\notin\mathbb{N}_0$ or

(i*) $a^*=0, \quad -c^*=0$ or
(v*) $a^*=1, \quad -c^*=1$.

REMARK 2.1. The twelve combinations of one of the possibilities (1), (i), (ii) or (iii) for a and c and one of the possibilities (1*), (i*) or (v*) for a^* and c^* just determine all the simple systems of Laurent polynomials satisfying (2.1), (2.1′) in the case that $A_3(x)=x^2(\lambda-x)$ with $\lambda\neq 0$. (Cf. [1] where in fact a characterization of "Sturm-Liouvillesche Systeme" has been given).

As we have to determine all the sequences $(V_n)_{n=0}^{\infty}$ which correspond to *orthogonal* systems $(Q_n)_{n=0}^{\infty}$ of Laurent polynomials only the following combinations of the possibilities for a and c and for a^* and c^* have to be examined

(2.10) $\quad a, -c, a^*, -c^* \notin \mathbb{N}_0,$

(2.11) $\quad a=c=a^*=c^*=0.$

This is easily verified using (1.2) which is equivalent to the orthogonality of the corresponding system $(Q_n)_{n=0}^{\infty}$. As an example we show that combination of

$$(1)\ a\notin \mathbb{N}_0,\ -c\notin \mathbb{N}_0 \quad \text{and} \quad (\mathrm{v}^*)\ a^*=-c^*=1$$

is in contradiction to the orthogonality of $(Q_n)_{n=0}^{\infty}$. Assume that (1) and (v*) are valid and that the V_n satisfy (1.2) with $f_n\neq 0$ and $g_n\neq 0$ for all n. Then

$$V_{2n}(x) = {}_2F_1(-2n, -a; -c-2n+1; x), \quad n\geqq 0$$

and

$$V_{2n+1}(x) = 1-\frac{2n+1}{2n-1}x+\xi_{2n+1}\left(\frac{-2n-1}{2n-1}x^{2n}+x^{2n+1}\right)$$

with $\xi_{2n+1}\neq 0$, $n\geqq 1$.

Comparing the coefficients of $x^3,\dots,x^{2n-2}$ in

(2.12) $\quad V_{2n+1}(x) = (1+g_{2n+1}x)V_{2n}(x)+f_{2n+1}xV_{2n-1}(x)$

for sufficiently large n, we get

$$-g_{2n+1} = \frac{(-2n+k)(-a+k)}{(-c-2n+1+k)(1+k)}, \quad k=2,3,\dots,2n-3.$$

As n is large enough this implies $g_{2n+1}=-1$, $a=-1$ and $c=1$, hence (2.12) takes the form

$$\begin{aligned}1-\frac{2n+1}{2n-1}x+\xi_{2n+1}\left(\frac{-2n-1}{2n-1}x^{2n}+x^{2n+1}\right) &= (1-x)(1+x+\cdots+x^{2n})+\\ &+f_{2n+1}x\left(1-\frac{2n-1}{2n-3}x+\xi_{2n-1}\left(\frac{-2n+1}{2n-3}x^{2n-2}+x^{2n-1}\right)\right)=\\ &=1-x^{2n+1}+f_{2n+1}x\left(1-\frac{2n-1}{2n-3}x+\xi_{2n-1}\left(\frac{-2n+1}{2n-3}x^{2n-2}+x^{2n-1}\right)\right).\end{aligned}$$

This implies $f_{2n+1}=0$. Contradiction.

The other combinations of the possibilities for a and c and for a^* and c^* which differ from (2.10) and (2.11) can be treated in a similar way.

Of the remaining situations (2.10) and (2.11) we first consider (2.11). Let us assume

$$a = c = a^* = c^* = 0.$$

Then by (4,i)

$$V_0(x) = 1 \quad \text{and} \quad V_n(x) = 1 + \xi_n x^n, \quad \xi_n \neq 0, \ n = 1, 2, \ldots.$$

It is a simple consequence of (1.2) that

$$\xi_n = \frac{\xi_1^2}{\xi_2} \left(\frac{\xi_2}{\xi_1} \right)^n, \quad n = 1, 2, \ldots,$$

so with $p = \xi_1^2/\xi_2$ we get $V_0(\xi_1 x/\xi_2) = 1$ and

$$V_n\left(\frac{\xi_1 x}{\xi_2} \right) = 1 + px^n, \quad n = 1, 2, \ldots.$$

The corresponding orthogonal system of Laurent polynomials is essentially the system of example 4. This system can be regarded as a degenerate form of case I since for the present values of the parameters a, c, a^*, c^* equations (2.6) and (2.6′) can be divided by $1 - x$.

Next we examine (2.10). Assume

$$a, -c, a^*, -c^* \notin \mathbb{N}_0$$

We will show that orthogonality of the corresponding orthogonal system of Laurent polynomials implies that $a = a^*$, $c = c^*$ and $a - c \notin \mathbb{N}_0$.

Let us assume that $a^* \neq a$.

By (1) (table 1) we have

$$\begin{aligned} &(2.13) \quad V_{2n}(x) = {}_2F_1(-2n, -a; -c - 2n + 1; x) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, 2, \ldots. \\ &(2.13') \quad V_{2n+1}(x) = {}_2F_1(-2n - 1, -a^*; -c^* - 2n; x) \end{aligned}$$

If we put $V_n(x) = \beta_0^{(n)} + \beta_1^{(n)} x + \cdots + \beta_n^{(n)} x^n$, $n = 0, 1, \ldots$, then $\beta_0^{(n)} = 1$ and for the f_n and g_n in (1.2) we find

$$(2.14) \quad \beta_n^{(n)} = g_1 g_2 \cdots g_n, \quad n = 1, 2, \ldots,$$

$$(2.15) \quad \beta_k^{(n)} = \beta_k^{(n-1)} + g_n \beta_{k-1}^{(n-1)} + f_n \beta_{k-1}^{(n-2)}, \quad k = 1, \ldots, n-1; \ n = 2, 3, \ldots.$$

With (2.13) and (2.13′) we get from (2.14) that

$$(2.16) \quad g_{2n} = -\frac{(-a)_{2n}(-c^* - 2n + 2)_{2n-1}}{(-c - 2n + 1)_{2n}(-a^*)_{2n-1}} \quad n = 1, 2, \ldots,$$

$$(2.16') \quad g_{2n+1} = -\frac{(-a^*)_{2n+1}(-c - 2n + 1)_{2n}}{(-c^* - 2n)_{2n+1}(-a)_{2n}} \quad n = 0, 1, 2, \ldots.$$

Elimination of f_n from (2.15) with $k=1$ respectively $k=2$ yields

$$(2.17)\quad (\beta_1^{(n-1)}-\beta_1^{(n-2)})g_n=\beta_2^{(n)}-\beta_2^{(n-1)}-\beta_1^{(n-2)}(\beta_1^{(n)}-\beta_1^{(n-1)}).$$

As we assumed $a^*\neq a$ we get from (2.17) and (2.13), (2.13′)

$$(2.18)\quad \lim_{n\to\infty} g_{2n}=\frac{a^*-a-1}{2}$$

and

$$(2.18')\quad \lim_{n\to\infty} g_{2n+1}=\frac{-a^*+a-1}{2}.$$

Since $\lim_{n\to\infty} g_{2n}g_{2n+1}=1$ this implies $(a^*-a)^2+3=0$, thus

$$(2.19)\quad a^*-a=\pm i\sqrt{3}.$$

Now in (2.15) we replace n by $2n$ and we eliminate f_{2n} from this equation for $k=1$ and $k=3$ and we get

$$(\beta_2^{(2n-1)}-\beta_2^{(2n-2)})g_{2n}=\beta_3^{(2n)}-\beta_3^{(2n-1)}-\beta_2^{(2n-2)}(\beta_1^{(2n)}-\beta_1^{(2n-1)}).$$

With (2.13), (2.13′) and (2.18) this implies, as $a^*\neq a$,

$$3(a^*+a-1)\frac{a^*-a-1}{2}=(a^*-a-1)(a^*+2a-2),$$

or equivalently

$$(a^*-a-1)(a^*-a+1)=0,$$

contradicting (2.19). Hence we have proved

$$(2.20)\quad a^*=a$$

In order to show that also $c^*=c$ we remark that by Stirling's formula we have

$$(2.21)\quad g_{2n}\approx\frac{\Gamma(c)}{\Gamma(c^*)}(2n)^{c^*-c}\quad \text{as } n\to\infty.$$

If in (2.15) we replace n by $2n$ and we eliminate f_{2n} from this formula for $k=1$ and $k=2n-1$, then we obtain

$$(2.22)\quad (\beta_{2n-2}^{(2n-1)}-\beta_{2n-2}^{(2n-2)})g_{2n}=\beta_{2n-1}^{(2n)}-\beta_{2n-1}^{(2n-1)}-\beta_{2n-2}^{(2n-2)}(\beta_1^{(2n)}-\beta_1^{(2n-1)}).$$

With (2.13) and (2.13′) this gives after multiplication by

$$\frac{(-c-2n+1)_{2n}}{(-a)_{2n}}$$

and using that $a^*=a$

$$(2.23)\quad \frac{(2n-1)c^*}{-a+2n-2}-\varphi_n g_{2n}=\frac{2nc}{-a+2n-1}-\frac{1}{g_{2n}}+\varphi_n\left(\frac{2n-1}{-c^*-2n+2}-\frac{2n}{-c-2n+1}\right)a$$

where

$$\varphi_n = \frac{(-c-2n+1)(-c-2n+2)}{(-a+2n-2)(-a+2n-1)}.$$

It follows easily from (2.23) that

$$\lim_{n\to\infty}\left(\varphi_n g_{2n} - \frac{1}{g_{2n}}\right) = c^* - c,$$

hence $\lim_{n\to\infty}\inf |g_{2n}|>0$ and $\lim_{n\to\infty}\sup |g_{2n}|<\infty$ and by (2.21) we have $\mathrm{Re}(c^*-c)=0$. Put $c^*-c=2i\eta$, $\eta\in\mathbb{R}$.

Then (2.21) becomes

$$g_{2n} \approx \frac{\Gamma(c)}{\Gamma(c^*)} e^{2i\eta\log 2n} \quad \text{as } n\to\infty.$$

Since for $\eta\neq 0$ and every positive integer n_0 the set $\{2n\,e^{k(\pi/\eta)}\colon n,k\in\mathbb{Z},\ n\geqq n_0\}$ is dense, in $(0,\infty)$ the set $\{e^{2i\eta\log 2n}\colon n\geqq n_0,\ n\in\mathbb{N}_0\}$ is dense in the unit circle in $\mathbb{C}$ if $\eta\neq 0$. On the other hand g_{2n} satisfies $\varphi_n g_{2n}^2 - 2\ell_n g_{2n} - 1 = 0$ where $\lim_{n\to\infty}\ell_n = i\eta$, so the zeros z_1 and z_2 of $z^2-2i\eta z-1$ are the only possible accumulation points of $(g_{2n})_{n=1}^{\infty}$. Since $z_1z_2\neq 0$ this is impossible, unless $\eta=0$. Hence we have proved that also

$$c^* = c$$

Thus in the case of (2.10) we have

$$V_n(x) = {}_2F_1(-n,-a;-c-n+1;x), \quad n=0,1,2,\dots.$$

From (2.15) with $k=1$ we easily obtain

$$f_n = (n-1)\frac{a-c-n+2}{(-c-n+1)(-c-n+2)}, \quad n=2,3,\dots.$$

Since orthogonality of the corresponding systems of Laurent polynomials implies $f_n\neq 0$, $n=2,3,\dots$ we have $a-c\notin\mathbb{N}_0$.

The corresponding system of Laurent polynomials is the system of example 1.

II $A_3(x)=x^2$

In this case we have

$$\begin{aligned}
&(2.24) \quad x^2Q_{2n}'' + x(\varrho x+\sigma)Q_{2n}' + A_1^{(2n)}Q_{2n} = 0,\\
&(2.24') \quad x^2Q_{2n+1}'' + x(\varrho^* x+\sigma^*)Q_{2n+1}' + A_1^{(2n+1)}Q_{2n+1} = 0
\end{aligned}
\qquad n=0,1,2,\dots.$$

Equating to zero the leading and trailing coefficients in (2.24) and (2.24′) and substituting $\sigma=-c+1$ and $\sigma^*=-c^*+2$ we get

$$(2.25) \quad x^2Q_{2n}'' + x(\varrho x-c+1)Q_{2n}' - n(\varrho x+c+n)Q_{2n} = 0,$$

$$(2.25') \quad x^2Q_{2n+1}'' + x(\varrho^* x-c^*+2)Q_{2n+1}' - [n\varrho^* x+(n+1)(c^*+n)]Q_{2n+1} = 0,$$
$$n=0,1,2,\dots.$$

For the corresponding polynomials $V_{2n}(x)=x^n Q_{2n}(x)$ and $V_{2n+1}(x)==x^{n+1}Q_{2n+1}(x)$ we have

$$(2.26)\quad xV_{2n}''+(\varrho x-c-2n+1)V_{2n}'-2n\varrho V_{2n}=0,$$

$$n=0,1,2,\dots.$$

$$(2.26')\quad xV_{2n+1}''+(\varrho^* x-c^*-2n)V_{2n+1}'-(2n+1)\varrho^* V_{2n+1}=0$$

If $\varrho=0$, then only if $c=0$ there exists a sequence of proper solutions of (2.26): $V_0(x)=1$ and $V_{2n}(x)=1+\xi_{2n}x^{2n}$, $\xi_{2n}\neq 0$, $n=1,2,\dots$. Similarly, if $\varrho^*=0$, we only have a sequence of proper solutions of (2.26′) if $c^*=0$: $V_{2n+1}(x)==1+\xi_{2n+1}x^{2n+1}$, $\xi_{2n+1}\neq 0$, $n=0,1,2,\dots$. If $\varrho\neq 0$ there are no sequences of proper solutions of (2.26) if $-c\in\mathbb{N}_0$ but if $-c\notin\mathbb{N}_0$ we have

$$V_{2n}(x)={}_1F_1(-2n;-c-2n+1;-\varrho x)\quad n=0,1,2,\dots.$$

In the case that $\varrho^*\neq 0$ a sequence of proper solutions of (2.26′) exists only if $-c^*\notin\mathbb{N}_0$. The solutions are

$$V_{2n+1}(x)={}_1F_1(-2n-1;-c^*-2n;-\varrho^* x)\quad n=0,1,2,\dots.$$

It follows easily from (1.2) that the only possibilities for orthogonal systems of simple Laurent polynomials $(Q_n)_{n=0}^\infty$ are the two combinations

$$(2.27)\quad \varrho=\varrho^*=c=c^*=0,$$

$$(2.28)\quad \varrho\neq 0,\ \varrho^*\neq 0,\ -c\notin\mathbb{N}_0,\ -c^*\notin\mathbb{N}_0.$$

In the case of (2.27) we have again the system of example 4. In the case of (2.28) we will show that orthogonality of the system $(Q_n)_{n=0}^\infty$ implies $\varrho=\varrho^*$ and $c=c^*$. Assume $\varrho\neq 0$, $\varrho^*\neq 0$, $-c\notin\mathbb{N}_0$, $-c^*\notin\mathbb{N}_0$. Again we write

$$V_n(x)=\beta_0^{(n)}+\beta_1^{(n)}x+\dots+\beta_{n-1}^{(n)}x^{n-1}+\beta_n^{(n)}x^n,\quad n=0,1,\dots.$$

Now we have

$$(2.29)\quad \beta_k^{(2n)}=\frac{(-2n)_k(-\varrho)^k}{k!\,(-c-2n+1)_k},\quad k=0,1,\dots,2n;\ n=0,1,2,\dots$$

$$(2.29')\quad \beta_k^{(2n+1)}=\frac{(-2n-1)_k(-\varrho^*)^k}{k!\,(-c^*-2n)_k},\quad k=0,1,\dots,2n+1;\ n=0,1,2,\dots.$$

It follows easily that

$$(2.30)\quad g_{2n}=\frac{\varrho^{2n}(-c^*-2n+2)_{2n-1}}{(\varrho^*)^{2n-1}(-c-2n+1)_{2n}}\quad n=1,2,\dots,$$

$$(2.30')\quad g_{2n+1}=\frac{(\varrho^*)^{2n+1}(-c-2n+1)_{2n}}{\varrho^{2n}(-c^*-2n)_{2n+1}}\quad n=0,1,2,\dots.$$

Assuming $\varrho\neq\varrho^*$ we get from (2.17), (2.29) and (2.29′) that $\lim_{n\to\infty}g_{2n}==\frac{1}{2}(\varrho^*-\varrho)$ and $\lim_{n\to\infty}g_{2n+1}=-\frac{1}{2}(\varrho^*-\varrho)$. As (2.30) and (2.30′) imply $\lim_{n\to\infty}g_{2n}g_{2n+1}=0$ we have a contradiction. Hence

$$\varrho=\varrho^*$$

and we may assume that

$$\varrho = \varrho^* = 1.$$

Now we show that also $c = c^*$. By Stirling's formula we have

$$(2.31)\quad 2ng_{2n} \approx -\frac{\Gamma(c)}{\Gamma(c^*)}(2n)^{c^*-c} \quad \text{as } n \to \infty.$$

From (2.22), (2.29) and (2.29′) we obtain

$$\frac{(2n-1)(-c^*)}{2n} - \psi_n 2ng_{2n} =$$

$$= -c - \frac{1}{2ng_{2n}} - \psi_n 2n\left(\frac{2n}{-c-2n+1} - \frac{2n-1}{-c^*-2n+2}\right)$$

where

$$\psi_n = \frac{(-c-2n+1)(-c-2n+2)}{(2n)^2}$$

and it follows that

$$(2.32)\quad \lim_{n\to\infty}\left(-\psi_n 2ng_{2n} + \frac{1}{2ng_{2n}}\right) = 2(c^*-c).$$

Using the same arguments as in case I one proves that (2.31) and (2.32) imply that

$$c^* = c.$$

Hence

$$V_n(x) = {}_1F_1(-n; -c-n+1; -x) \quad n = 0, 1, 2, \ldots$$

and the corresponding system of Laurent polynomials is the system of example 2.

III $A_3(x) = x^3$

Now start from

$$(2.33)\quad x^3 Q''_{2n} + x(\varrho x + \sigma) Q'_{2n} + A_1^{(2n)} Q_{2n} = 0,$$

$$(2.33')\quad x^3 Q''_{2n+1} + x(\varrho^* x + \sigma^*) Q'_{2n+1} + A_1^{(2n+1)} Q_{2n+1} = 0 \qquad n = 0, 1, 2, \ldots.$$

With $\varrho = -a+1$ and $\varrho^* = -a^*+2$ we get from (2.33) and (2.33′)

$$(2.34)\quad x^3 Q''_{2n} + x[(-a+1)x + \sigma] Q'_{2n} - n[(-a+n)x - \sigma] Q_{2n} = 0,$$

$$(2.34')\quad x^3 Q''_{2n+1} + x[(-a^*+2)x + \sigma^*] Q'_{2n+1} - [n(-a^*+n+1)x - (n+1)\sigma^*] Q_{2n+1} = 0$$

and the corresponding polynomials $V_{2n}(x) = x^n Q_{2n}(x)$ and $V_{2n+1}(x) = x^{n+1} Q_{2n+1}(x)$ satisfy

$$(2.35) \quad x^2 V''_{2n} + [(-a-2n+1)x+\sigma] V'_{2n} + 2naV_{2n} = 0$$

$$n=0,1,2,\ldots.$$

$$(2.35') \quad x^2 V''_{2n+1} + [(-a^*-2n)x+\sigma^*] V'_{2n+1} + (2n+1)a^* V_{2n+1} = 0$$

If $\sigma=0$ then there is a sequence of proper solutions of (2.35) if and only if $a=0$. These solutions are of the form $V_0(x)=1$ and $V_{2n}(x)=1+\xi_{2n}x^{2n}$, $\xi_{2n}\neq 0$, $n=1,2,\ldots$.

If $\sigma^*=0$ then only for $a^*=0$ we have proper solutions V_{2n+1}, $n=0,1,2,\ldots$, of (2.35′) and $V_{2n+1}(x)=1+\xi_{2n+1}x^{2n+1}$, $\xi_{2n+1}\neq 0$, $n=0,1,2,\ldots$. In the case that $\sigma\neq 0$ there are no sequences of proper solutions of (2.35) if $a\in\mathbb{N}_0$ but if $a\notin\mathbb{N}_0$ we have the proper solutions

$$V_{2n}(x) = {}_2F_0\left(-2n, -a; -\frac{x}{\sigma}\right), \quad n=0,1,2,\ldots.$$

Similarly, if $\sigma^*\neq 0$ we only have sequences $(V_{2n+1})_{n=0}^{\infty}$ of proper solutions of (2.35′) if $a^*\notin\mathbb{N}_0$. For $a^*\notin\mathbb{N}_0$ we have

$$V_{2n+1}(x) = {}_2F_0\left(-2n-1, -a^*; -\frac{x}{\sigma^*}\right), \quad n=0,1,2,\ldots.$$

Again it follows from (1.2) that only the combinations

$$(2.36) \quad \sigma=\sigma^*=a=a^*=0,$$

$$(2.37) \quad \sigma\neq 0,\ \sigma^*\neq 0,\ a\notin\mathbb{N}_0,\ a^*\notin\mathbb{N}_0$$

may lead to an orthogonal system $(Q_n)_{n=0}^{\infty}$ of simple Laurent polynomials satisfying (2.33) and (2.33′).

As in the previous case (2.36) gives the system of example 4. We show that in the case of (2.37) we must have $\sigma=\sigma^*$ and $a=a^*$ when the sequence $(Q_n)_{n=0}^{\infty}$ is orthogonal. Assume

$$\sigma\neq 0,\ \sigma^*\neq 0,\ a\notin\mathbb{N}_0,\ a^*\notin\mathbb{N}_0$$

With

$$V_n(x) = \beta_0^{(n)} + \beta_1^{(n)}x + \cdots + \beta_{n-1}^{(n)}x^{n-1} + \beta_n^{(n)}x^n, \quad n=0,1,2,\ldots$$

we have

$$(2.38) \quad \beta_k^{(2n)} = \frac{(-2n)_k(-a)_k}{k!\,(-\sigma)^k} \quad k=0,1,\ldots,2n;\ n=0,1,2,\ldots,$$

$$(2.38') \quad \beta_k^{(2n+1)} = \frac{(-2n-1)_k(-a^*)_k}{k!\,(-\sigma^*)^k} \quad k=0,1,\ldots,2n+1;\ n=0,1,2,\ldots,$$

and

$$(2.39) \quad g_{2n} = \frac{(\sigma^*)^{2n-1}(-a)_{2n}}{\sigma^{2n}(-a^*)_{2n-1}} \quad n=1,2,\ldots,$$

$$(2.39') \quad g_{2n+1} = \frac{\sigma^{2n}(-a^*)_{2n+1}}{(\sigma^*)^{2n+1}(-a)_{2n}} \quad n=0,1,2,\ldots.$$

With (2.38) and (2.38′) we get from (2.17)

$$(2.40)\quad \left(\frac{(-2n+1)a^*}{\sigma^*}+\frac{(2n-2)a}{\sigma}\right)g_{2n}=\frac{(-2n)_2(-a)_2}{2\sigma^2}-\frac{(-2n+1)_2(-a^*)_2}{2(\sigma^*)^2}+$$

$$+\frac{(2n-2)a}{\sigma}\left(\frac{-2na}{\sigma}+\frac{(2n-1)a^*}{\sigma^*}\right),$$

$$(2.40')\quad \left(\frac{-2na}{\sigma}+\frac{(2n-1)a^*}{\sigma^*}\right)g_{2n+1}=\frac{(-2n-1)_2(-a^*)_2}{2(\sigma^*)^2}-\frac{(-2n)_2(-a)_2}{2\sigma^2}+$$

$$+\frac{(2n-1)a^*}{\sigma^*}\left(\frac{(-2n-1)a^*}{\sigma^*}+\frac{2na}{\sigma}\right).$$

Using (2.38) and (2.38′) we obtain from (2.22)

$$\frac{(2n-1)\sigma^*}{-a^*+2n-2}-\theta_n g_{2n}=\frac{2n\sigma}{-a+2n-1}-\frac{1}{g_{2n}}-\theta_n\left(\frac{(2n-1)a^*}{\sigma^*}-\frac{2na}{\sigma}\right),$$

where $\theta_n=\sigma^2/(-a+2n-2)(-a+2n-1)$, and it follows that

$$(2.41)\quad \lim_{n\to\infty}\left(\theta_n g_{2n}-\frac{1}{g_{2n}}\right)=\sigma^*-\sigma.$$

Suppose $a^*/\sigma^*\neq a/\sigma$. Then (2.40) and (2.40′) imply

$$\lim_{n\to\infty}\frac{g_{2n}}{2n}=\frac{\frac{1}{2}\left[\frac{(-a)_2}{\sigma^2}-\frac{(-a^*)_2}{(\sigma^*)^2}\right]-\frac{a}{\sigma}\left[\frac{a}{\sigma}-\frac{a^*}{\sigma^*}\right]}{\frac{a}{\sigma}-\frac{a^*}{\sigma^*}}$$

and

$$\lim_{n\to\infty}\frac{g_{2n-1}}{2n}=\frac{\frac{1}{2}\left[\frac{(-a)_2}{\sigma^2}-\frac{(-a^*)_2}{(\sigma^*)^2}\right]-\frac{a^*}{\sigma^*}\left[\frac{a}{\sigma}-\frac{a^*}{\sigma^*}\right]}{\frac{a}{\sigma}-\frac{a^*}{\sigma^*}},$$

hence

$$\lim_{n\to\infty}\left(\frac{g_{2n}}{2n}-\frac{g_{2n-1}}{2n}\right)=\frac{a^*}{\sigma^*}-\frac{a}{\sigma}.$$

Since $\theta_n g_{2n}=1/g_{2n-1}$ and $\lim_{n\to\infty}(2n)^2\theta_n=\sigma^2$ it follows that

$$\lim_{n\to\infty}2n\left(\theta_n g_{2n}-\frac{1}{g_{2n}}\right)=\sigma^2\left(\frac{a^*}{\sigma^*}-\frac{a}{\sigma}\right)$$

But this means that the limit in (2.41) is zero, so

$$\sigma=\sigma^*\quad\text{if}\quad\frac{a^*}{\sigma^*}\neq\frac{a}{\sigma}.$$

Suppose now that $a^*/\sigma^* = a/\sigma$. Then (2.40) and (2.40′) take the forms

$$g_{2n} = \frac{a^*-1}{2\sigma^*}(2n-1)(2n-2) - \frac{a-1}{2\sigma}2n(2n-1) + \frac{a}{\sigma}(2n-2),$$

$$g_{2n+1} = \frac{a-1}{2\sigma}2n(2n-1) - \frac{a^*-1}{2\sigma^*}(2n+1)2n + \frac{a}{\sigma}(2n-1),$$

hence

$$\lim_{n\to\infty}\frac{g_{2n}}{(2n)^2} = \frac{1}{2}\left(\frac{1}{\sigma}-\frac{1}{\sigma^*}\right) \quad \text{and} \quad \lim_{n\to\infty}\frac{g_{2n+1}}{(2n)^2} = \frac{1}{2}\left(\frac{1}{\sigma^*}-\frac{1}{\sigma}\right),$$

and also

$$\lim_{n\to\infty}\frac{g_{2n}g_{2n+1}}{(2n)^4} = -\frac{1}{4}\left(\frac{1}{\sigma}-\frac{1}{\sigma^*}\right)^2 \tag{2.42}$$

Since by (2.39) and (2.39′) we also have

$$\lim_{n\to\infty}\frac{g_{2n}g_{2n+1}}{(2n)^2} = \frac{1}{(\sigma^*)^2}$$

it follows from (2.42) that $\sigma=\sigma^*$. Thus if $a^*/\sigma^*=a/\sigma$, then $\sigma=\sigma^*$.

Now we know that $\sigma=\sigma^*$ we may assume $\sigma=\sigma^*=1$.

Under this last assumption it can be proved as in case I that $a^*=a$. So

$$V_n(x) = {}_2F_0(-n,-a;-x) \quad n=0,1,2,\dots$$

and the corresponding orthogonal system of Laurent polynomials is the system of example 3.

Now we have finished the treatment of the cases I, II and III we have completed the proof of the following theorem.

THEOREM 2.1. If $(Q_n)_{n=0}^{\infty}$ is a simple orthogonal system of Laurent polynomials such that

$$\begin{aligned} &A_3Q''_{2n} + A_2Q'_{2n} + A_1^{(2n)}Q_{2n} = 0, \\ &A_3Q''_{2n+1} + A_2^*Q'_{2n+1} + A_1^{(2n+1)}Q_{2n+1} = 0 \end{aligned} \quad n=0,1,2,\dots$$

where $A_3A_2A_2^*$ are fixed polynomials, independent of n with $A_3\not\equiv 0$, $\deg A_2 = \deg A_2^*$ and $A_1^{(2n)}, A_1^{(2n+1)}$ are polynomials depending on n but with $\deg A_1^{(2n)} \leqq 1$ and $\deg A_1^{(2n+1)} \leqq 1$, $n=0,1,2,\dots$, then there is a nonzero $t\in\mathbb{C}$ such that $(Q_n(tx))_{n=0}^{\infty}$ is a classical orthogonal system of Laurent polynomials, i.e. $(Q_n(tx))_{n=0}^{\infty}$ is one of the systems of examples 1, 2, 3 or $(Q_n(tx))_{n=0}^{\infty}$ is the system of example 4.

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